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CONTENTS:

MSS. intended for publication and books, etc., intended for review should be sent to the responsible editor, Professor J. McKeen Cattell, Garrison-on-Hudson, N. Y.

SUPPLEMENTARY REPORT ON NON-EUCLIDEAN GEOMETRY.

WHEN at the Columbus meeting of the American Association I had the honor of making a 'Report on Non-Euclidean Geometry,' it was mentioned that my own 'Bibliography of Hyper-space and Non-Euclidean Geometry,' in the *American Journal of Mathematics* (1878), giving 81 authors and 174 titles, when reprinted in the collected works of Lobachevski (Kazan, 1886) gives 124 authors and 272 titles; while Roberto Bonola had just given (1899) a 'Bibliography of the Foundations of Geometry in Relation to Non-Euclidean Geometry,' containing over 350 titles with some repetitions.

Bonola in 1900 finished a second part of this bibliography, in which the single section headed 'Historical, Critical and Philosophical Writings' gives 96 authors and 150 titles. It thus becomes very evident that a most important function of your reporter is the selection of what writings to bring forward for especial mention as of paramount importance and typical of the main stream of advance.

In the Columbus report I particularly stressed the work of two authors whom I brought forward together and to whom I devoted about a quarter of that report.

The report first appeared in SCIENCE for October 20, 1899, and you may imagine that it was reassuring when on October 22 (old style), 1900, the Commission of the Physico-Mathematical Society of Kazan found the scientific merits of the works of these authors, A. N. Whitehead and Wm. Killing equal for the great Lobachevski prize and had to decide between them by the drawing of lots.

In his report on the work of Whitehead, Sir Robert Ball says of the 'Universal Algebra':

"Several other writers, to whom of course Mr. Whitehead makes due acknowledgment, have approached the study of non-Euclidean geometry by the aid of Grassmann's methods, but the systematic and most instructive development of the subject in book VI. is, I believe, new, as are also many of the results obtained.

"The superiority of Whitehead's methods appears to lie in the two following features:

"1°. That he can treat n dimensions by practically the same formulæ as those used for two or three dimensions.

"In this I think he has made a considerable advance upon the methods, ingenious and beautiful as some of them no doubt are, which have been used by previous investigators.

"2°. The various kinds of space, parabolic, hyperbolic and elliptic (of two kinds), present themselves in Whitehead's methods quite naturally in the course of the work, where they appear as the only alternatives when certain assumptions have been made.

"Moreover the results have been obtained in such a way that it is easy for the reader to develop for one of the other spaces properties treated out in full for one space only.

"The book deserves in the highest degree the attention of the student of modern mathematical methods, and it marks so great an advance that it is, in my judgment, well worthy of the important prize in view of which this report is prepared.

"Mr. Whitehead's memoir on geodesics in elliptic space appears to me to indicate great power in dealing with a very difficult problem. I believe it to be of much importance, as the geodesics in the generalized space conceptions had been but little studied."

In the corresponding report on the work of Killing, Professor Engel, of Leipsic, says of the 'Grundlagen der Geometrie':

"This work is, from the first to the last page, a justification and detailed development of the circle of ideas which we are accustomed to understand under the expression 'non-Euclidean geometry.'"

"Already so many preliminary questions have been settled," said Killing in the preface to his first volume, "that the final solution can be hoped for at a not too distant time."

"These words written in 1893," says Engel, "have meanwhile most recently (1899) found a highly striking confirmation in many directions through Hilbert's investigations.

"The geometries possible with the Euclidean, namely the Lobachevski-Bolyaian, the Riemannian and the elliptic, Killing develops, each for itself, in Euclidean way up to a certain grade.

"Also it should not be forgotten that Killing was the first, who (1879, *Crelles Journal*, Bd. 83) made clear the difference between the Riemannian and the elliptic space (or as he calls it, the Polar form of the Riemannian).

"The fourth section treats the Clifford-Klein space-forms, in whose investigation Killing himself has taken a conspicuous part (by a work in Bd. 39 of the *Mathematische Annalen*, 1891). The great importance of these space-forms rests upon this, that they show with especial clearness, what a mighty difference it makes whether we, from the beginning, assume the geometric axioms as valid for space as a whole or merely for an every way bounded piece of space. In the first case we obtain, besides the Euclidean, only the three previously mentioned non-Euclidean space-forms.

"In the second case appears also a manifoldness, at present not yet dominated, of different space forms.

"The treatment of continuity and the ratio-idea in Euclid gives occasion for a nearer investigation of the so-called Archimedes' Axiom.

"Finally, as the first attempt to illuminate in conjunction all the different questions which have grouped themselves about the problem mentioned, and to collect all the means, which numerous mathematicians, and not least the author himself, have made for solving the problem, this work will for long retain its value.

"That precisely the founding of geometry since the appearance of this book has been advanced in a wholly unexpected way by Hilbert, cannot lessen Killing's merit. His work remains still by far the best means for mastering the researches which have appeared in this realm up to 1898." These interesting extracts I take from the Russian pamphlet just issued at Kazan and furnished me by my friend Professor Vasiliev.

In his paper 'Ueber Nicht-Euklidische und Linien-Geometrie' (Greifswald, 1900), Professor E. Study voices a profound truth when he says: "The conception of geometry as an experimental science is only one among many possible, and the standpoint of the empiric is as regards geometry by no means the richest in outlook. For he will not, in his one-sidedness, justly appreciate the fact that in manifold and often surprising ways the mathematical sciences are intertwined with one another, that in truth they form an indivisible whole.

"Although it is possible and indeed highly desirable, that each separate part or theory be developed independently from the others and with the instrumentalities peculiar to it, yet whoever should disregard the manifold interdependence of the different parts, would deprive himself of one of the most powerful instruments of research.

"This truth, really self-evident yet often not taken to heart, applied to Euclidean and non-Euclidean geometry, leads to the somewhat paradoxical result that, among conditions to a more profound understanding of even very elementary parts of the Euclidean geometry, the knowledge of the non-Euclidean geometry cannot be dispensed with."

That the world has caught one deduction from this deep idea, is shown by the fact of the almost simultaneous appearance of two text-books, manuals for class use, to make universally attainable this necessary condition for any thorough understanding of any geometry, even the most elementary; two intended, available popular treatises on this ever more essential non-Euclidean geometry.

One of these, just being issued by G. Carré et C. Naud, 3 rue Racine, Paris, is 'La géométrie non Euclidienne,' by P. Barbarin, professor at Bordeaux, a place made sacred for non-Euclideans by the memory of Hoüel. How great and practical is the interest of this book can be gathered from the headings of its chapters.

I. 'General and historical considerations.' How the non-Euclidean doctrine was born and gradually developed.

II. 'Euclid's definitions and postulates.' Study of the rôle that they play in the principles of geometry.

Simple and elementary exposé of the three geometries after the method of Saccheri.

III. 'Distance as fundamental notion.' The definitions of the straight and the plane according to Cauchy. The works of M. De Tilly.

IV. 'General geometry in the plane and in space.' Résumé of the principal general propositions.

V. 'Trigonometry.' Elementary demonstration, after Gérard and Mansion, of the formulas for triangles and quadrilaterals.

VI. 'Measurement of areas and volumes.'

VII. 'The contradictors of the non-Euclidean geometry.' The principal objections made against the non-Euclidean geometry. Answers to be made thereto.

VIII. 'Physical geometry.' How we might attempt to find out if the physical world is not Euclidean; how angles and distances could be measured with a much greater approximation, for example, angles with an error much less than $\frac{1}{100}$ of a second.

A brief article by Professor Barbarin, 'On the utility of studying non-Euclidean geometry,' which appears in the May (1901) number of Professor Cristoforo Alasia's new Italian journal *Le Matematiche*, shows that Hoüel had reached the weighty insight which we have quoted from Study, namely, that knowledge of non-Euclidean geometry is essential for any mastery of Euclidean geometry.

Says Barbarin:

"I. The question of the source of the theory of parallels has been one of the most interesting scientific preoccupations of the century; it has made to flow torrents of books, and given theme to remarkable works. Thanks to the theorems of Legendre, to the discoveries of the two Bolyai, of Lobachevski and of Riemann; thanks to the original researches of Beltrami and of Sophus Lie, of Poincaré, Flye Ste. Marie, Klein, De Tilly, etc., we cannot any more be mistaken as to the true scope of the celebrated proposition which bears the name of Postulate of Euclid.

"1°. This is not in any way contained in the classic definitions of the straight and the plane.

"2°. This is, among three hypotheses equally admissible, and which cannot all be rejected, only the most simple.

"Is it perhaps chance alone which gave to the great Greek geometer the choice of his system of geometry? or did he perceive, at least in part, the difficulties and the greater theoretic complication of the other two? We shall never know with certainty.

"But in the presence of his work, so perfect and so rigorous, one thing, however, appears not to be doubtful: the place which he assigned to his proposition, the enunciation which he gave of it, attest that this proposition had to his eyes only the value of an hypothesis; otherwise he would have formulated it in other terms and would have attempted to demonstrate it.

"The ideas of Lobachevski and of Riemann were diffused only very slowly. They were so, above all, thanks to the translations of Hoüel.

"This scientist, whose activity and power of work were prodigious, could not resist the desire to master all the European languages, with the aim of being able to read in their original text, and then make known to his contemporaries the most celebrated mathematical works.

"He admired Lobachevski, whom he surnamed the *modern Euclid*, and in his course professed at the scientific faculty of Bordeaux, he did not let pass any occasion to put him in evidence.

"II. Hoüel was persuaded that the knowledge of the non-Euclidean geometry is indispensable for thoroughly mastering the mechanism of the Euclidean geometry.

"Despite its paradoxical form, this idea is most just.

"General geometry or *metageometry* contains in fact a great number of propositions common to all the systems, and which ought to be enunciated in the same terms in each of these. If the general proposition can be demonstrated in terms general for these, such should be preferred, even if, to attain this, it be necessary to subject the ordinary form to some modification. To cite only one example, we take the convex quadrilateral inscribed in a circle.

"In Euclidean geometry, *the sum of two opposite angles is constant and equal to two right angles;* in non-Euclidean geometry *this sum is variable.* Notwithstanding this, the two forms may be reconciled, since in both cases *the sum of two opposite angles equals that of the other two,* and this is sufficient for a convex quadrilateral to be inscriptible.

"Confronting the proposition with that which concerns the circumscribed quadrilateral, we put in full light a correlation which, *à priori,* ought evidently to exist.

"This correlation, which is the very heart of general geometry, and which does not always appear in the ordinary geometry with the same clearness, can be utilized for finding new properties of the figures.

"*Example: Every conic is the locus of the points such that the sum of the tangents from these drawn to two circles is constant;* every conic then will also be *the curve envelope of the straights which cut two given circles under angles of which the sum is constant.* (Excellent problem for investigating directly.)

"III. Is it expedient to associate the non-Euclidean geometry with instruction, and in what measure?

"If we treat of higher instruction, with ardor we respond affirmatively.

"In the courses of higher geometry of the universities the names of Bolyai, Lobachevski, Riemann have their assigned place, and there are still divers unexplored domains on the road which these scientists have opened.

"In so far as it refers to secondary instruction, the question is more delicate. The programs of preparatory courses at the high schools contain all, or almost all, special mathematics and spherical geometry.

"It would not be then a great inconvenience to there make opportunely a discrete allusion to general geometry: on the contrary, the attention of the students and their critical spirit would be held awake by the necessity of investigating if such proposition which is expounded to them is of order particular or general.

"At least two indispensable conditions should be satisfied; it is requisite:

"1°. *That in all the books put in the hands of the students, the hypothetical and wholly factitious character of the Euclidean postulate be put well into relief.*

"In my classes I have recourse with success to the simple procedure which follows, and which I recommend. Take the straight AB and the two equal perpendiculars AB, BD: the angles ACD, BDC are equal, and may be right, acute or obtuse. But whichever be the one among these three hypotheses which we assume for this particular quadrilateral, we must conserve it for *all* the other like quadrilaterals. We choose the system of geometry in which these are right angles, and which corresponds to the Euclidean hypothesis.

"2°. That the invertibility of the postulate of Euclid be completely given up in all the demonstrations in which it can be done without and where nevertheless it is wrongly used.

"See, for example, the theorem on the face angles of a trihedral or polyhedral angle.

"We should recognize that great advances have been made in these latter years in the sense indicated.

"If the ideas of general geometry tend to become popularized, the honor of it is due above all to the periodicals which have given their hospitality, and in special manner to *Mathesis,* so ably edited by our excellent confrère, P. Mansion of Ghent.

"In the course of the last eight or ten years this journal has published numerous articles on Metageometry, written with as much competence as good sense. We counsel their perusal."

It will be seen from our quotation, that Professor Barbarin bases his exposition on the method of Saccheri as the simplest.

The same is true in the other new text-book, 'Manning's Non-Euclidean Geometry.' (Boston, Ginn & Co., 1901, 8vo, pp. v+95.)

Saccheri's first proposition is (*American Mathematical Monthly*, June, 1894, Vol. I., p. 188):

"If two equal straights, *AC*, *BD*, make with the straight *AB* angles equal toward the same parts: I say the angles at the join *CD* will be mutually equal."

On the next page is "Proposition II. The quadrilateral *ABCD* remaining the same, the sides *AB*, *CD* are bisected in points *M* and *H*. I say the angles at the join *MH* will be on both sides right."

Professor Manning paraphrases these two together on page 5.

"If two equal lines in a plane are erected perpendicular to a given line, the line joining their extremities makes equal angles with them and is bisected at right angles by a third perpendicular erected midway between them."

Under the heading 'Definitions,' Saccheri says: "Since (from our first) the straight joining the extremities of equal perpendiculars standing upon the same straight (which we will call base), makes equal angles with these perpendiculars, three hypotheses are to be distinguished according to the species of these angles.

"And the first, indeed, I will call hypothesis of right angle; the second, however, and the third I will call hypothesis of obtuse angle, and hypothesis of acute angle." This Manning paraphrases as follows, under the heading 'The Three Hypotheses':

"The angles at the extremities of two equal perpendiculars are either right angles, acute angles, or obtuse angles, at least for restricted figures. We shall distinguish the three cases by speaking of them as the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle respectively."

Saccheri's Proposition III. is: "If two equal straights, *AC*, *BD*, stand perpendicular to any straight, *AB*: I say the join *CD* will be equal, or less, or greater than *AB*, according as the angles at *CD* are right, or obtuse, or acute."

This Manning paraphrases as follows: "The line joining the extremities of two equal perpendiculars is, at least for any restricted portion of the plane, equal to, greater than or less than the line joining their feet in the three hypotheses respectively."

In the same way is paraphrased Saccheri's Prop. IV., the converse of III.

Saccheri's corollary about quadrilaterals with three right angles is given by Manning on page 12.

Saccheri's Prop. V. is: "The hypothesis of right angle, if even in a single case it is true, always in every case it alone is true."

In giving this, Manning has: 'If the hypothesis of *a* right angle,' etc., evidently a slip for his usual *the* right angle. Of course the Latin original, of which I have, so far as I know, the only copy on this continent, has no article.

Prop. VI. and Prop. VII. are combined by Manning on p. 13.

Prop. IX. is reproduced on p. 14.

Prop. X. is given on p. 9.

In Prop. XI. Saccheri with the hypothesis of right angle demonstrates the celebrated Postulatum of Euclid, thus showing that his hypothesis of right angle is the ordinary Euclidean geometry.

Manning says, p. 27: "The three hypotheses give rise to three systems of geometry, which are called the parabolic, the hyperbolic and the elliptic geometries. They are also called the Geometries of Euclid, of Lobachevski, and of Riemann." Now Saccheri in his demonstration of Prop. XI. makes, almost in the words of Archimedes, an assumption, introduced by the words 'it is manifest,' which we now call, for convenience, Archimedes' Axiom. In his futile attempts at demonstrating the parallel-postulate, Legendre set forth two theorems, called Legendre's theorems on the angle-sum in a triangle. They are:

1. In a triangle the sum of the three angles can never be greater than two right angles.

2. If in any triangle the sum of the three angles is equal to two right angles, so is it in every triangle.

In addition to assuming the infinity or two-sidedness of the straight, in his proofs of these theorems Legendre uses essentially the Archimedes Axiom. Thence he gets that the angle-sum in a triangle equaling two right angles is equivalent to the parallel-postulate, all of which is really what Saccheri gave a century before him, now just reproduced by Barbarin and Manning, as before by De Tilly. Even Hilbert in his 'Vorlesung ueber Euklidische Geometrie' (winter semester, 1898–99), for a chance to see Dr. von Schafer's Authographie of which I am deeply grateful to Professor Bosworth, gives the following five theorems and then says: "Finally we remark, that it seems as if each of these five theorems could serve precisely as *equivalent of the Parallel Axiom.*" They are

1. The sum of the angles of a triangle is always equal to two right angles.

2. If two parallels are cut by a third straight, then the opposite (corresponding) angles are equal.

3. Two straights, which are parallel to a third, are parallel to one another.

4. Through every point within an angle less than a straight angle, I can always draw straights which cut both sides [not perhaps their prolongations].

5. All points of a straight have from a parallel the same distance.

But since then a wonderful discovery has been made by M. Dehn.

It was known that Euclid's geometry could be built up without the Archimedes axiom. So arises the weighty question: *In such a geometry do the Legendre theorems necessarily hold good?*

In other words: Can we prove the Legendre theorems without making use of the Archimedes axiom?

This is the question which, at the instigation of Hilbert, was taken up by Dehn.

His article was published July 10, 1900 (*Mathematische Annalen*, 53 Band, pp. 404–439).

Dehn was able to demonstrate Legendre's second theorem without using any postulate of continuity, a remarkable advance over Saccheri, Legendre, De Tilly.

But his second result is far more remarkable, namely, that Legendre's first theorem is indemonstrable without the Archimedes axiom.

To prove this startling position, Dehn constructs a new non-Euclidean geometry, which he calls a 'non-Legendrean' geometry, in which through every point an infinity of parallels to any straight can be drawn, yet in which nevertheless the angle sum in every triangle is greater than two right angles.

Thereby is the undemonstrability of the first Legendre theorem without the help of the Archimedes axiom proven.

Dehn then discusses the connection between the three different hypotheses relative to the sum of the angles [the three hypotheses of Saccheri, Barbarin, Manning] and the three different hypotheses relative to the number and existence of parallels.

He reaches the following remarkable propositions:

From the hypothesis that through a given point we can draw an infinity of parallels to a given straight it follows, if we exclude the Archimedes axiom, *not* that the sum of the angles of a triangle is less than two right angles, but on the contrary that this sum may be (*a*) greater than two right angles, (*b*) equal to two right angles.

The first case (*a*) is proven by the existence of the non-Legendrean geometry.

To demonstrate the second case (*b*), Dehn constructs a geometry wherein the parallel-axiom does not hold good, and wherein nevertheless are verified all the theorems of Euclidean geometry; the sum of the angles of a triangle is equal to two right angles, similar triangles exist, the extremities of equal perpendiculars to a straight are all situated on the same straight, etc.

As Dehn states this result: There are non-Archimedian geometries, in which the parallel-axiom is not valid and yet the angle-sum in every triangle is equal to two right angles.

Such a geometry he calls '*semi-Euclidean.*'

Therefore, it follows that none of the theorems, the angle-sum in the triangle is two right angles, the equidistantial is a straight, etc., can be considered as equivalent to the parallel-postulate, and that Euclid in setting up the parallel-postulate hit just the right assumption.

This is a marvelous triumph for Euclid.

Finally Dehn arrives at this surprising theorem:

From the hypothesis that there are no parallels, it follows that the sum of the angles of a triangle is greater than two right angles.

Thus the two non-Euclidean hypotheses about parallels act in a manner absolutely different with regard to the Archimedes Axiom.

The different results obtained may now be tabulated thus:

The angle-sum in the triangle is:	Through a given point we can draw to a straight:		
	No parallel.	One parallel.	An infinity of parallels.
$> 2R$	Elliptic geometry	(Impossible)	Non-Legendrean geometry
$= 2R$	(Impossible)	Euclidean geometry	Semi-Euclidean geometry
$< 2R$	(Impossible)	(Impossible)	Hyperbolic geometry

Riemann, Helmholtz and Sophus Lie founded geometry on an analytical basis in contradistinction to Euclid's pure synthetic method.

They elected to conceive of space as a manifold of numbers. In the Columbus report is an account of the Helmholtz-Lie investigation of the essential characteristics of space by a consideration of the movements possible therein.

This is notably simplified if we suppose given *à priori* the graphic concepts of straight and plane, and admit that movement transforms a straight or a plane into a straight or respectively a plane. Killing determines analytically the three types of projective groups, but the same results are reached in a way geometric and purely elementary by Roberto Bonola in a beautiful little article entitled, 'Determinazione, per via geometrica, dei tre tipi di spazio: Iperbolico, Ellittico, Parabolico (*Rendiconti del Circolo Matematico di Palermo*, Tomo XV., pp. 56–65, April, 1901).

In 1833 was published in London the fourth edition of an extraordinary book (3d Ed., 1830) by T. Perronet Thompson of Queen's College, Cambridge, with the following title:

'Geometry without Axioms.'

"Being an attempt to get rid of Axioms and Postulates; and particularly to establish the theory of parallel lines without recourse to any principle not grounded on previous demonstration.

"To which is added an appendix containing notices of methods at different times proposed for getting over the difficulty in the 'Twelfth Axiom of Euclid.'" 8vo, pp. x + 148. This dissects most brilliantly twenty-one methods of getting rid of Euclid's postulate; so brilliantly that it deserves to be reprinted and could scarcely be improved upon. Then, nothing daunted by the failure of every one else of whom he has ever heard, the brave Thompson adds one of his own, which perhaps he also afterward impaled upon the point of his keen dissecting scalpel, for he lived long and prospered. In 1865 De Morgan, whose unknown letters to Sylvester I had the pleasure of publishing in the *Monist*, writes:

"With your note came an acknowledgement from General Perronet Thompson, B.A. of 1802, and Fellow of Queen's before he was an ensign. And he works at acoustics as hard as ever he did at the Corn Laws."

But even in 1833, had he but known it, the question of two thousand years, as to whether Euclid's Parallel-Axiom could be deduced, had been settled at last by the creation and indeed publication, by Bolyai, and also by Lobachevski, of a geometry in which it is flatly contradicted.

The newly created methods, which thus settled this old, old question, give entirely new views concerning the investigation of axioms in general; and this diamond mine has been masterfully preempted by Hilbert, of Göttingen. His wonderful 'Grundlagen der Geometrie' is ablaze with gems from this non-Euclidean mine.

After Bolyai and Lobachevski, Hilbert's closest forerunner is Friedrich Schur, of Karlsruhe. One of the most fundamental advances of this decade is the strict rigorous reduction of the comparison of areas to the comparison of sects.

This was first given on January 23, 1892, by Schur before the Dorpater Naturforscher-Gesellschaft.

The account printed in Russia in the society's *Proceedings*, a *Referat* given by Schur, is of course almost inaccessible, nor is this inaccessibility much lessened for us by the fact that it has been translated into Italian (Per. di Mat., VIII., p. 153).

The essence of the matter is the proof that, a certain sect being taken as the measure of the area of a triangle, the *sum* of these sects is *the same* for any set of triangles into which a given polygon can be cut, and so gives a sect which may be taken as the measure of the area of the polygon. The *Referat* begins as follows:

"On the surface content of plane figures with straight boundaries, by Friedrich Schur.

"So simple a problem as the measuring of plane figures with straight boundaries as it seems from the literature to me accessible, has not yet been set forth with the rigor and purity of method herein possible.

"Not to mention the introduction of endless processes, still general magnitude-axioms are used unjustifiably, which are only then immediately clear when these magnitudes are straight sects, their comparison therefore capable of being made by superposition.

"Such a general magnitude-theorem, which is used in all text-books of elementary mathematics known to me in proving the theorem of the equal area of two parallelograms with common base and equal altitude, is, *e. g.*, this, that the subtraction of equal magnitudes from equal magnitudes gives again equal magnitudes.

"If the sides of the two parallelograms lying opposite the common base have a piece or at least a point in common, then the two parallelograms can at once be cut into parts such that each part of the one parallelogram corresponds to a part congruent to it of the other parallelogram.

"On the contrary, if those two sides have no point in common, then it has been believed that this method of proof for the equality of area, simple and standing upon a sharp definition, must be renounced, and it has been replaced, as is known, by this, that each of the two parallelograms is represented as the difference between the same trapez and one of two congruent triangles.

"But before the measurement of plane surfaces by sects has been attained, which just first becomes possible through the theorem to be proven, the application of the above magnitude-theorem is justified by nothing.

"We must therefore throw away this method of proof, and that so much the more, as in every case each of two parallelograms with common base and equal altitude in very simple way comprehensible to every scholar can be so cut into a number of parts that to each part of the one parallelogram corresponds a part congruent to it of the other.

"One may find that, *e. g.*, set forth in 'Stoltz's Vorlesungen' ueber allgemeine Arithmetik, I. Theil (Leipzig, 1885), S. 75 ff.

"We can still somewhat simplify this method, and lessen the number of parts. Draw, namely, through each of the two end-points next one another of the sides lying opposite the common base, parallels to the sides of the other parallelogram, and prolong these to the two outer of the sides not parallel to the base. The join of the two end-points so obtained is then parallel to the base, and cuts from the two parallelograms two new parallelograms which without anything further are divided into triangles every two congruent to one another.

"If then the sides opposite the common base of the remaining parallelograms again have no common point, then we proceed just so with them, and come thus, after a finite number of repetitions, to a pair of parallelograms, to which the customary procedure can be applied.

"If the distance of those two neighboring end-points of the sides opposite the base is greater than the $n$fold of the base, on the other hand at the highest equal to the $(n+1)$fold of the base, then is each parallelogram cut into a trapez (respectively triangle), three triangles and n parallelograms, and each of such parts of the one parallelogram corresponds to a part congruent to it of the other." Now it so happens that I myself had reached this method and published it seven years before Schur in my 'Elements of Geometry' (John Wiley & Sons, New York). It may be described more concisely as taking away pairs of congruent triangles each with base equal to the common base of the two parallelograms and sides respectively parallel to their other pairs of sides, until we have left two parallelograms to which the customary dissection into a triangle and trapezoid will apply, to finish with congruent parts.

But this demonstration, though the very simplest possible, yet postulates the Archimedes axiom, though neither I myself, in 1885, nor Schur, seven years later, in 1892, said a word about this assumption. However, before 1898 Schur became conscious that elementary geometry can be built up without the Archimedes axiom. He states this in the preface to his remarkable 'Lehrbuch der analytischen Geometrie' (Leipzig, Veit & Comp., 1898), referring to his article 'Ueber den Fundamentalsatz der projectiven Geometrie' *Math. Annalen*, Bd. 51), where he proves the theorems of Desargues and of Pascal without using either the parallel postulate or the axiom of Archimedes, proving that the ordinary sect-calculus can be built up independently of number-measure and the Archimedean postulate.

Professor Anne Bosworth, of Rhode

Island, has followed this up by actually constructing in her doctor's dissertation at Göttingen (1900), under Hilbert, a sect-calculus independent of the parallel-axiom.

This is a beautiful piece of non-Euclidean geometry, and is, so far as I know, the first feminine contribution to our fascinating subject.

In 1899 appeared Hilbert's 'Grundlagen der Geometrie,' in which the remarkable contributions of Schur are all retouched by a master hand.

In Schur's proof of the Pascal theorem the space axioms are used. Hilbert replaces them by the parallel-axiom, thus proving Pascal as a theorem of plane Euclidean geometry.

Schur makes a sect-calculus, and shows that the theory of proportion can be founded without the introduction of the difficult idea of the irrational number. He indicates that this could be done without the Archimedes axiom.

Hilbert actually does it.

Schur proves for the first time the fundamental theorem for a rigorous treatment of area.

Hilbert simplifies this proof, and proceeds to treat this whole subject without the Archimedes axiom, making here the new distinction between flächengleich and inhaltsgleich.

Two polygons are said to have *equal surface* when they can be resolved into a finite number of triangles congruent in pairs.

Two polygons are said to have *equal content* if it is possible to add to them polygons of equal surface, so that the two new compound polygons have equal surface.

Thus Euclid only tried to treat *equal content*, and Hilbert is here a return to the great Greek.

The intense interest in all these unexpected developments is voiced in a handsome volume: 'Questioni riguardanti la geometria elementare' (Bologna, 1900, 8vo, pp. vii + 532), edited by Federigo Enriques, who has been chosen to contribute the part on the foundations of geometry to the great German Encyclopædia of the Mathematical Sciences, and who contributes the first article (28 pages) to this Italian work. It is entitled 'On the Scientific and Didactic Importance of the Questions which Relate to the Principles of Geometry.'

The whole book may be properly described as an outcome of the non-Euclidean geometry, but more specifically, the longest of the fourteen articles which make it up is by Bonola: 'On the Theory of Parallels and on the non-Euclidean Geometries' (80 pages, 26 figures).

The first fifty of his eighty pages are devoted to an historico-critical exposition; the last thirty to general theory, hyperbolic geometry, elliptic geometry. Though the article was published only last year, it is in certain respects antiquated. The proofs freely use the Archimedes postulate, without saying anything more about it than I did in 1885, that is, nothing at all. His § 7 is headed 'Postulates Equivalent to the Postulate of Euclid,' and gives those adopted by Proclos, Wallis, Bolyai Farkas, Carnot, Legendre, Laplace, Gauss. But now we know that all these men failed in attempting to rival the choice of Euclid. Their axioms are not the equivalent of his immortal postulate.

In this section the name Legendre is misspelled, and in § 5 Bonola says, "The attempts of Legendre for the demonstration of the Euclidean hypothesis, published in the various editions of the 'Elements' of Euclid, which appear under his name," etc.

Of course Legendre never published any edition of Euclid. It was on the contrary Legendre's geometry which cursed the subject with that definition, "A straight line is the shortest distance between two points," which still disgraces the beautifully illustrated book of Phillips and Fisher of Yale.

Again, in § 12 Bonola misquotes in a very important particular the title of the only thing Bolyai János ever published, his renowned appendix, in which title, instead of 'Johanne Bolyai de eadem,' Bonola has 'Johanne Bôlyai de Bôlya.' Again in § 8 Bonola is still expressing the hope that the examination of the unedited manuscripts of Gauss may show some ground for the pretence that Gauss had some part, however minute, in the creations of Bolyai, Lobachevski and Riemann. But these manuscripts have already been most sympathetically edited by Professor Paul Staeckel, their publication making a goodly quarto, in a review of which for SCIENCE under the heading 'Gauss and the non-Euclidean Geometry,' I find they only strengthen the already existing demonstration that neither of the creators of the non-Euclidean geometry owed even the minutest fraction of an idea or suggestion to Gauss.

This is reproven by the correspondence of Gauss and Bolyai Farkas, so sumptuously published in royal quarto by the Hungarian Academy of Science, edited by Staeckel and Franz Schmidt, chiefly valuable for its references to the immortal boy, Bolyai János, of whom unfortunately no portrait exists.

And now a word in conclusion.

Thinking is important for life. So much so that evolution in thinking has dominated all other evolution. In all thinking enters a creative element. There is not any pure receptivity. Nothing can be described except in terms of a precreated theory. The business of science is the making of these theories, and the continual remaking and bettering of these theories. The higher races of mankind, and chiefly the Greeks, created and elaborated a scheme for dominating what a popular terminology calls the facts and laws presented by the spatial relations of things.

This scheme was only one of an indefinite number of possible schemes, but as coordinated and systematized by a great constructive genius, Euclid, the first professor of mathematics at the University of Alexandria, it proved so efficient, so effective for life, that all educated men accepted it as part of their common equipment.

Though it promises no heaven, though it threatens no hell, though it mentions no angels, no devils, yet Euclid's elements of geometry, simply as conveying a necessary instrument for the conduct of civilized life, has appeared in more than one thousand four hundred different editions [Professor Riccardi: Saggio di una bibliografia Euclidea (Bologna, 'Memorie' (5), I., 1890)].

Euclid gave to educated mankind a common language for description of the spatial, a common mental basis for thought about extension. Euclid's geometry is a certain theory for a specific natural science, a mental construction to explain, to master, to communicate or transmit, and to prophesy certain physical phenomena, the spatial or extensive phenomena. Therefore, the body of its doctrine is a system of theorems deduced in a logical way from certain unproven and in part absolutely and finally indemonstrable assumptions. Such a one is the world-renowned parallel-postulate, which is absolutely incapable of being proved in any way whatsoever, mental or physical, speculative or experimental, deductive or inductive. Therefore, to substitute for it a contradiction of it, in Euclid's scheme of fundamental assumptions, is to get with certainty another equally logical theory to do all that Euclid's geometry has ever done.

Of such systems each may throw light on the other, each may possess special advantages for particular applications.

But more than that: three such systems

used simultaneously may be able to accomplish what no one of them could do. This is beautifully illustrated in a theory communicated to me by F. W. Frankland, using a cosmic medium in which small regions of elliptic and hyperbolic space alternate, given a strain toward parabolic space which produces an elasticity or resilience simulating the properties with which physicists have endowed their hypothetical ether.

GEORGE BRUCE HALSTED.

UNIVERSITY OF TEXAS.

THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

SECTION H, ANTHROPOLOGY.

THE effect of environment on the success of a meeting was well demonstrated at Denver. Local interest in the Section of Anthropology, fostered by the Colorado Cliff Dwellings' Association, had reached such a pitch even in advance of the opening session that the small room originally intended for the Section was abandoned for one with a seating capacity of 200. This large room was converted into a bazaar of rare aboriginal ceramics, Navajo blankets, basketry and pictures of Indian scenes by a committee from the Cliff Dwellings Association consisting of Mrs. J. D. Whitmore, Mrs. G. T. Sumner and Mrs. W. S. Peabody, all of Denver.

The meeting was memorable for sustained interest. The attendance was unprecedented, averaging at least 150 for the morning sessions; the afternoon audiences were also large.

Section H was organized on Monday morning, August 26, after the adjournment of the General Session, in accordance with the provisions of the constitution. The officers for the Denver meeting were as follows:

Vice-President, J. Walter Fewkes.
Secretary, George Grant MacCurdy.

Sectional Committee: A. W. Butler, vice-president, Section H, 1900; Frank Russell, secretary, Section H, 1900; J. Walter Fewkes, vice-president, Section H, 1901; George Grant MacCurdy, secretary, Section H, 1901; Mrs. M. L. D. Putnam, Frank W. Blackmar, G. A. Dorsey.

Member of Council, W J McGee. *Member of General Committee*, Mrs. W. S. Peabody.

Retiring Vice-president Butler's address, entitled, 'A Notable Factor in Social Degeneracy,' was delivered Monday afternoon. It was printed in SCIENCE of September 20.

The titles of papers presented before the Section are accompanied by brief abstracts in so far as it has been possible to secure these from the authors.

1. 'Exhibit of Curves of Speech': E. W. SCRIPTURE.

An exhibit of a series of plates containing the curves of vibration traced from a gramophone plate containing Rip Van Winkle's Toast spoken by Joseph Jefferson. In the absence of the author, the paper was presented by Mr. MacCurdy. It will be printed in Scripture's 'Elements of Experimental Phonetics.'

2. 'Influences of Racial Characteristics on Socialization': FRANK W. BLACKMAR.

Racial characteristics are the great barriers that prevent a complete socialization of the human race; the race idea, or consciousness on the part of two groups of people that they are different in origin and structure is a detriment to perfect social union; a transition from the old family, or ethnographic, status to the modern, or demographic, society has been exceedingly difficult; the race idea has been the hindering cause in the progress of democracy; the historical examples of the social difficulties of Greece, Rome the Iroquois tribes and the Aztec federation; the difficulties of socialization appear in the development of homogeneous society in large cities; difficulties arising from an attempt to socialize widely divergent races; when common marriage relations are pro-